4.5: Indeterminate Forms and L'Hopital's Rule

L'Hopital's Rule was first discovered by Johann Bernoulli (again) and allows the user to calculate limits of fractions when both the numerator and denominator approach 0 or ∞.

It is named after Guillaume de L'Hopital who wrote an introductory differential calculus text where it first appeared in print.

Def^n: An indeterminate form of a quotient or product appears as

$$\frac{0}{0}, \quad \frac{\infty}{\infty}, \quad \infty \cdot 0, \quad \infty - \infty, \quad 0^0 \text{ or } 1^\infty .$$

Theorem: (L'Hopital's Rule)

(a) Suppose that $f(a) = g(a) = 0$ and f, g are diff. near $x = a$.

If $g'(a) \neq 0$ then $\displaystyle \lim_{x \to a} \frac{f(x)}{g(x)} = \lim_{x \to a} \frac{f'(x)}{g'(x)}$.

(b) Suppose that $\displaystyle \lim_{x \to a^-} \frac{f(x)}{g(x)} = \frac{\infty}{\infty}$ then really

$$\lim_{x \to a^-} \frac{f(x)}{g(x)} = \lim_{x \to a^-} \frac{f'(x)}{g'(x)} .$$

Remark: If you have an indeterminate form and L'Hopital's rule gives you another indeterminate form, just use it again until it works.

Ex (1):

(a) $\lim\limits_{x\to 0} \dfrac{3x-\sin x}{x} = \lim\limits_{x\to 0} \dfrac{3-\cos x}{1} = 2$ (can be checked w/ other rules)

(b) $\lim\limits_{x\to 0} \dfrac{\sqrt{1+x}-1}{x} = \lim\limits_{x\to 0} \dfrac{\frac{1}{2\sqrt{1+x}}}{1} = \dfrac{1}{2}$

(c) $\lim\limits_{x\to 0} \dfrac{x-\sin x}{x^3} \underset{\text{L'Hopital}}{=} \lim\limits_{x\to 0} \dfrac{1-\cos x}{3x^2} \underset{\text{L'Hopital}}{=} \lim\limits_{x\to 0} \dfrac{\sin x}{6x} \underset{\text{L'Hopital}}{=} \lim\limits_{x\to 0} \dfrac{\cos x}{6} = \dfrac{1}{6}.$

(d) Be careful! As soon as you get a non-indeterminate form you must stop!

$\lim\limits_{x\to 0} \dfrac{1-\cos x}{x+x^2} = \dfrac{0}{0} \underset{\text{L'Hopital}}{\Longrightarrow} \lim\limits_{x\to 0} \dfrac{\sin x}{1+2x} = \dfrac{0}{1} = 0.$

But if we do it again $\lim\limits_{x\to 0} \dfrac{\cos x}{2} = \dfrac{1}{2}$ (Not Correct)

Remark: In some cases you get different 1-sided limits.

(e) $\lim\limits_{x\to 0^+} \dfrac{\sin x}{x^2} \underset{\text{L'Hopital}}{=} \lim\limits_{x\to 0^+} \dfrac{\cos x}{2x} = \infty$

but $\lim\limits_{x\to 0^-} \dfrac{\sin x}{x^2} \underset{\text{L'Hopital}}{=} \lim\limits_{x\to 0^-} \dfrac{\cos x}{2x} = -\infty.$

Ex 2:

(a) $\displaystyle\lim_{x\to\pi/2} \frac{\sec x}{1+\tan x} = \frac{\infty}{\infty} \underset{\text{L'Hopital}}{\Longrightarrow} \lim_{x\to\pi/2} \frac{\sec x\tan x}{\sec^2 x} = \lim_{x\to\pi/2} \sin x = 1.$

(b) $\displaystyle\lim_{x\to\infty} \frac{\ln x}{2\sqrt{x}} = \frac{\infty}{\infty} \underset{\text{L'Hopital}}{\Longrightarrow} \lim_{x\to\infty} \frac{1/x}{1/\sqrt{x}} = \lim_{x\to\infty} \frac{1}{\sqrt{x}} = 0$

(c) $\displaystyle\lim_{x\to\infty} \frac{e^x}{x^2} \underset{\text{L'Hopital}}{=} \lim_{x\to\infty} \frac{e^x}{2x} \underset{\text{L'Hopital}}{=} \lim_{x\to\infty} \frac{e^x}{2} = \infty.$

L'Hopital's Rule also works for $\infty\cdot 0$ and $\infty - \infty$.

But you rewrite to look as before. $\overset{??}{\frac{\infty}{\infty}}$ $\overset{??}{\frac{0}{0}}$

Ex (3):

(a) $\cancel{\displaystyle\lim_{x\to\infty} x\cdot\sin\left(\frac{1}{x}\right) \underset{\text{L'Hopital}}{=} \lim_{x\to\infty} 1\cdot\frac{-\cos x}{x^2}}$ Show this but it doesn't work!!

$\displaystyle\lim_{h\to 0^+} \frac{1}{h}\cdot\sinh \underset{\text{L'Hopital}}{=} \lim_{h\to 0^+} \frac{\cosh}{1} = 1$ (we have seen this).

(b) $\displaystyle\lim_{x\to 0^+} \sqrt{x}\cdot\ln x = \lim_{x\to 0^+} \frac{\ln x}{1/\sqrt{x}} \underset{\text{L'Hopital}}{=} \lim_{x\to 0^+} \frac{1/x}{-\frac{1}{2}x^{3/2}} = \lim_{x\to 0^+} -2\sqrt{x} = 0$

(c) $\displaystyle\lim_{x\to 0} \frac{1}{\sin x} - \frac{1}{x} = \lim_{x\to 0} \frac{x-\sin x}{x\sin x} \underset{\text{L'Hopital}}{=} \lim_{x\to 0} \frac{1-\cos x}{\sin x + x\cos x} = \frac{0}{0}$

L'Hopital Again $\Rightarrow \displaystyle\lim_{x\to 0} \frac{\sin x}{2\cos x - x\sin x} = \frac{0}{2} = 0.$

Indeterminate Powers 0^0 and 1^∞ and ∞^0.

In this case, we consider L'Hopital's rule on $\ln f(x)$.

Ex(4):

(a) $\lim\limits_{x\to 0^+}(1+x)^{1/x} = 1^\infty$ so $\lim\limits_{x\to 0^+}\ln((1+x)^{1/x}) = \lim\limits_{x\to 0^+}\frac{1}{x}\cdot\ln(1+x) = \frac{0}{0}$

$\underset{\text{L'Hopital}}{\Longrightarrow} \lim\limits_{x\to 0^+}\frac{1/_{1+x}}{1} = 1$. Thus $\lim\limits_{x\to 0^+}(1+x)^{1/x} = e^1 = e$. (Seen before).

(b) $\lim\limits_{x\to\infty} x^{1/x} = \infty^0$. So $\lim\limits_{x\to\infty}\ln(x^{1/x}) = \lim\limits_{x\to\infty}\frac{1}{x}\ln(x) = \frac{\infty}{\infty}$

$\underset{\text{L'Hopital}}{\Longrightarrow} \lim\limits_{x\to\infty}\frac{1/x}{1} = \frac{0}{1} = 0$.

The proof of L'Hopital's Rule is based on Cauchy's MVT which is a generalization of the standard MVT.

Cauchy's MVT: f, g satisfy the hypotheses of MVT and $g'(x)\neq 0$ on (a,b). Then $\exists\, c\in(a,b)$ such that

$$\frac{f'(c)}{g'(c)} = \frac{f(b)-f(a)}{g(b)-g(a)}.$$

Proof of L'Hopital: Enough to consider $\frac{0}{0}$ case.
Consider interval $[a,x]$. Then by Cauchy's MVT, $\frac{f'(c)}{g'(c)} = \frac{f(x)-f(a)}{g(x)-g(a)}$ $(c\in(a,x))$.
But $f(a)=g(a)=0$. So $\frac{f'(c)}{g'(c)} = \frac{f(x)}{g(x)}$. As $x\to a^+$, $c\to a^+$. The result follows.